Set $\Psi_0 : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \to \mathbb{C}$ restriction of Ψ

Set $\Omega = \sum_\mu I_\mu \otimes I_\mu$ and

$$[\Omega^{(ij)} \Psi_0](\xi_1, \ldots, \xi_n)$$

$$= \sum_\mu \Psi_0(\xi_1, \ldots, I_\mu \xi_i, \ldots, I_\mu \xi_j, \ldots, \xi_n)$$

Recall: $[(X \otimes t^{-1})^{(i)} \Psi](\xi_1, \ldots, \xi_n)$

$$= \sum_{j, j \neq i} (z_i - z_j)^{-1} \Psi(\xi_1, \ldots, X\xi_j, \ldots, \xi_n) \qquad (*)$$

<u>Proposition 2:</u>

If a multilinear form $\Psi : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \to \mathbb{C}$ belongs to the space of conformal blocks $\mathcal{H}(p_1, \ldots, p_n ; \lambda_1, \ldots, \lambda_n)$, then the restriction $(L_{-1}^{(i)} \Psi)_0$ of $L_{-1}^{(i)} \Psi : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \to \mathbb{C}$ on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ is given by

$$(L_{-1}^{(i)} \Psi)_0 = \sum_{j, j \neq i} \frac{\Omega^{(ij)} \Psi_0}{z_i - z_j}$$

Furthermore, we have

(1) $(L_n^{(i)} \Psi)_0 = 0, \ n > 0$ (2) $(L_0^{(i)} \Psi)_0 = \Delta_{\lambda_i} \Psi_0$

where Δ_{λ_i} is eigenvalue of L_0 on V_{λ_i}.

<u>Proof:</u>

For $v \in V_{\lambda_j} \subset H_{\lambda_j}$ we have

$$L_{-1} v = \frac{1}{k+2} \left(\sum_\mu I_\mu \otimes t^{-1} \cdot I_\mu \right) v$$

Combining with (*) we get

$$\sum_\mu \Psi(\xi_1, \dots, (I_\mu \otimes t^{-1} I_\mu)\xi_i, \dots, \xi_n)$$

$$= \sum_{j, j \neq i} \sum_\mu (z_i - z_j)^{-1} \Psi_0(\xi_1, \dots, I_\mu \xi_i, \dots, I_\mu \xi_j, \dots, \xi_n)$$

$$= \sum_{j, j \neq i} \frac{\Omega^{(ij)} \Psi_0}{z_i - z_j}$$

Equations (1) and (2) follow directly from definition of Sugawara operators. $\square$

Combining Theorem 1 and Proposition 2, gives

<u>Theorem 2:</u>

Let Ψ be a horizontal section of the conformal blocks bundle $\mathcal{E}_{\lambda_1 \dots \lambda_n}$. Then the restriction Ψ_0 satisfies

$$\frac{\partial \Psi}{\partial z_i} = \frac{1}{k+2} \sum_{j, j \neq i} \frac{\Omega^{(ij)} \Psi_0}{z_i - z_j}, \quad 1 \leq i \leq n$$

"Knizhnik-Zamolodchikov equation"

∇ is called "KZ connection"

Proposition 3:

Let Ψ be a horizontal section of the conformal block bundle $\mathcal{E}_{\lambda_1 \cdots \lambda_n}$. Then Ψ_0 satisfies

$$\sum_{i=1}^{n} z_i^{r} \left(z_i \frac{\partial}{\partial z_i} + (r+1) \Delta_{\lambda_i} \right) \Psi_0 = 0$$

for $r = -1, 0, 1$

Proof:

Invariance of Ψ_0 under diagonal action of $\mathfrak{g}$ gives

$$\sum_{j=1}^{n} \Omega^{(ij)} \Psi_0 = 0, \quad 1 \leq i \leq n$$

$\longrightarrow$ Taking sum over i gives:

$$\sum_{j \neq i} \Omega^{(ij)} \Psi_0 = - \sum_{j=1}^{n} \Omega^{(jj)} \Psi_0$$

$$\Rightarrow \sum_{1 \leq i < j \leq n} \Omega^{(ij)} \Psi_0 = -(K+2) \sum_{j=1}^{n} \Delta_{\lambda_j} \Psi_0 \qquad (**)$$

by using that $\Omega^{(ij)} = \Omega^{(ji)}$ and $\Omega^{(jj)}$ Casimir

$$\Rightarrow \sum_{i=1}^{n} z_i^{r+1} \frac{\partial \Psi_0}{\partial z_i} = \frac{1}{K+2} \sum_{i \neq j} \frac{z_i^{r+1} \Omega^{(ij)} \Psi_0}{z_i - z_j}$$

for $r = -1$: rhs $= 0$ $\left(\Omega^{(ij)} \text{ sym}, \frac{1}{z_i - z_j} \text{ anti-sym.} \right)$

for $r = 0$: rhs $= \frac{1}{K+2} \sum_{1 \leq i < j \leq n} \Omega^{(ij)}$

for $r=1$: rhs $= \frac{1}{K+2} \sum_{1\le i<j\le n} \frac{z_i^2 \Omega^{(ij)}}{z_i - z_j} + \frac{z_j^2 \Omega^{(ij)}}{z_j - z_i}$

$$= \frac{1}{K+2} \sum_{1\le i<j\le n} \frac{\cancel{(z_i - z_j)}(z_i + z_j)}{\cancel{z_i - z_j}} \Omega^{(ij)}$$

Use $(**)$, then claim follows. $\square$

Proposition 4:

Let Ψ be a horizontal section of the conformal block bundle $\mathcal{E}_{\lambda_1 \dots \lambda_n}$. Under a Möbius trf.

$$w_j = \frac{a z_j + b}{c z_j + d}, \quad 1 \le j \le n,$$
$$a, b, c, d \in \mathbb{C}, \ ad - bc = 1,$$

Ψ_0 behaves as

$$\Psi_0(z_1, \dots, z_n) = \prod_{j=1}^{n} (c z_j + d)^{-2\Delta_{\lambda_j}} \Psi_0(w_1, \dots, w_n).$$

Proof:

The case $r=-1$ of Prop. 3 shows thas Ψ_0 is invariant under

$$w_j = z_j + c, \quad c \in \mathbb{C}, \quad 1 \le j \le n.$$

In the $r=0$ case, $\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$ is the so-called "logarithmic derivative" and Prop. 3 gives

$$\Psi_0(w_1, \dots, w_n) = \alpha^{-\Delta_{\lambda_1} - \dots - \Delta_{\lambda_n}} \Psi_0(z_1, \dots, z_n).$$

Möbius trfs. of type $f_\varepsilon(z) = \frac{z}{-\varepsilon z + 1}$ are called special conformal trfs. and we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Psi_0\big(f_\varepsilon(z_1), \dots, f_\varepsilon(z_n)\big) = \sum_{j=1}^{n} z_j^2 \frac{d}{dz_j} \Psi_0(z_1,\dots,z_n)$$

$$\text{Prop. 3} \Rightarrow \Psi_0^{f_\varepsilon} = \prod_{j=1}^{n} (-\varepsilon z_j + 1)^{2\Delta_j} \Psi_0(z_1,\dots,z_n)$$

Since group of Möbius trfs. is generated by above 3 trfs., the claim follows. □

§ 5.1 Solutions of KZ equation

Fix finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ together with representations

$$\rho_j : \mathfrak{g} \longrightarrow \mathrm{End}(V_j), \quad 1 \le j \le n.$$

Denote by $\{I_\mu\}$ orthonormal basis of $\mathfrak{g}$ with respect to Cartan-Killing form and set

$$\Omega = \sum_\mu I_\mu \otimes I_\mu$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$,

$$\Omega = \frac{1}{2} H \otimes H + E \otimes F + F \otimes E$$

The element $C = \sum_\mu I_\mu I_\mu$ in the universal enveloping algebra $U(\mathfrak{g})$ is called "Casimir elem."

We have $\Omega = \frac{1}{2}\left(\Delta C - C \otimes 1 - 1 \otimes C\right)$ (1)

where $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is comultiplication

(e.g. $\Delta(I_\mu I_\mu) = 2 I_\mu \otimes I_\mu + I_\mu I_\mu \otimes 1 + 1 \otimes I_\mu I_\mu$)

Next, consider logarithmic differential 1-forms

$$\omega_{ij} = d\log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}, \quad i \neq j,$$

defined on $\mathrm{Conf}_n(\mathbb{C})$.

$\longrightarrow$ satisfy quadratic relations

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0, \quad i<j<k$$

"Arnold relations" (exercise)

Let $\phi : V_1 \otimes V_2 \otimes \cdots \otimes V_n \longrightarrow \mathbb{C}$ be a multilinear form. We denote by $\Omega^{(ij)}\phi$ the multi-linear form

$$(\Omega^{(ij)}\phi)(v_1 \otimes \cdots \otimes v_n) = \sum_\mu \phi(v_1 \otimes \cdots \otimes \rho_i(I_\mu) v_i \otimes \cdots \otimes \rho_j(I_\mu) v_j \otimes \cdots \otimes v_n)$$

for $v_1 \otimes \cdots \otimes v_n \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Then the KZ equation is given by

$$\frac{\partial \Phi}{\partial z_i} = \frac{1}{\kappa} \sum_{j, j \neq i} \frac{\Omega^{(ij)} \Phi}{z_i - z_j} \qquad (*)$$

where κ is a non-zero complex parameter and $\Phi(z_1, \cdots, z_n)$ is defined over $\mathrm{Conf}_n(\mathbb{C})$

with values in $\mathrm{Hom}_{\mathbb{C}}(V_1 \otimes V_2 \otimes \cdots \otimes V_n, \mathbb{C})$

Now, we put

$$\omega = \frac{1}{K} \sum_{1 \le i < j \le n} \Omega^{(ij)} \omega_{ij}$$

$\longrightarrow$ (*) becomes $d\hat{\Phi} = \omega \hat{\Phi}$.

<u>Lemma 1:</u>

The above $\Omega^{(ij)}$, $1 \le i \ne j \le n$, satisfy the following relations:

1. $\Omega^{(ij)} = \Omega^{(ji)}$
2. $[\Omega^{(ij)} + \Omega^{(jk)}, \Omega^{(ik)}] = 0$, i, j, k distinct
3. $[\Omega^{(ij)}, \Omega^{(kl)}] = 0$, i, j, k, l distinct.

<u>Proof:</u>

Relations 1 and 3 are clear. We show relation 2. Consider the case $n=3$.

Casimir element lies in center of $U(\mathfrak{g})$:

$$[\Delta(C), \Delta(X)] = 0$$

in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ for any $X \in U(\mathfrak{g})$. Thus

$$[\Delta(C) \otimes 1, \sum_{\mu} \Delta(I_\mu) \otimes I_\mu] = 0$$

Together with $\Omega = \frac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C)$, we get

$$[\Omega^{(12)}, \Omega^{(13)} + \Omega^{(23)}] = 0$$

since $C\otimes 1\otimes 1$ and $1\otimes C\otimes 1$ lie in ~~center~~ center of $U(\mathfrak{g})\otimes U(\mathfrak{g})\otimes U(\mathfrak{g})$. Similarly for other index choices. $\square$

Lemma 2:

We have $\omega\wedge\omega = 0$

Proof:

$$\omega\wedge\omega = \frac{1}{K^2}\sum_{i<j,\,k<l}\left[\Omega^{(ij)},\Omega^{(kl)}\right]\omega_{ij}\wedge\omega_{kl}$$

The Arnold relation then gives

$$\sum_{i<j,\,k<l}\left[\Omega^{(ij)},\Omega^{(kl)}\right]\omega_{ij}\wedge\omega_{kl}$$

$$= \sum_{i<j<k}\Big(\left[\Omega^{(ij)}+\Omega^{(jk)},\Omega^{(ik)}\right]\omega_{ij}\wedge\omega_{ik}$$

$$+\left[\Omega^{(ij)}+\Omega^{(ik)},\Omega^{(jk)}\right]\omega_{ij}\wedge\omega_{jk}\Big)$$

$$+\sum_{\{i,j\}\cap\{k,l\}=\emptyset}\left[\Omega^{(ij)},\Omega^{(kl)}\right]\omega_{ij}\wedge\omega_{kl},$$

which vanishes by Lemma 1. $\square$

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